MA 681, Spring 2011

Assignment 4

Cauchy's Integral Formula. Taylor Series

I prefer that you submit this assignment by Wednesday, April 21st. However, if you are somehow delayed, take your time (just don't get overwhelmed by homeworks piling up.)

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

(1) Evaluate the integral

$$\frac{1}{w\pi i}\int_L \frac{ze^z}{(z-a)^3}dz,$$

where $a \in I(L)$. (Hint: use formula for derivative of Cauchy's integral.).

(2) *Cauchy's Integral Formula for unbounded domain* Let L be a closed rectifiable simple curve, traversed in the counterclockwise direction. Let $f(z)$ be an analytic function on a domain G, where $L \cup E(L)$ is contained in G. (That is, $f(z)$ is analytic in the neighborhood of L and *outside* of L, but not necessarily everywhere inside.) In particular, $f(z)$ is analytic at infinite, that is $f(1/w)$ is analytic at $w = 0$. Suppose that

$$\lim_{z\to\infty} f(z) = A.$$

Prove that

$$\frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\zeta - z}d\zeta = A, \quad \text{if } z \in I(L),$$

and

$$\frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\zeta - z}d\zeta = -f(z) + A, \quad \text{if } z \in E(L).$$

(Hint. Make the substitution $z = 1/w$, $\zeta = 1/\eta$.

(3) Find the radius of convergence of the following series.
(a) $\sum_{n=0}^{\infty} n^k z^n$, $k = 0, 1, 2, \ldots$ (b) $\sum_{n=0}^{\infty} n^n z^n$, (c) $\sum_{n=0}^{\infty} 2^n z^n$,
(d) $\sum_{n=0}^{\infty}(3 + (-1)^n)^n z^n$, (e) $\sum_{n=0}^{\infty} \cos inz^n$, (e) $\sum_{n=0}^{\infty} \frac{n^k}{n!} z^n$,
(f) $\sum_{n=0}^{\infty} \frac{n!}{n^n} z^n$, $\sum_{n=0}^{\infty} \frac{2^n}{n!} z^n$, (g) $\sum_{n=0}^{\infty}(n + a^n)z^n$,, (h) $\sum_{n=0}^{\infty} z^{n^2}$,
(Hint: you can use the fact that $\sqrt[n]{\frac{n!}{n^n}} \to \frac{1}{e}$, $n \to \infty$.)

(4) Given that radius of convergence of the power series

$$\sum_{n=0}^{\infty} c_n z^n$$

is R $(0 < R < \infty)$, what is the radius of convergence of each the following series:
(a) $\sum_{n=0}^{\infty} n^k c_n z^n$, $k = 1, 2, \ldots$, (b) $\sum_{n=0}^{\infty}(2^n - 1)c_n z^n$, (c) $\sum_{n=0}^{\infty} \frac{c_n}{n^n} z^n$,
(d) $\sum_{n=0}^{\infty} c_n^k z^n$, $k = 0, 1, 2, \ldots$.

(5) Expand each of the following functions as a Taylor series at $z = 0$:
(a) $\sin z^2$, (b) $\frac{1}{ax+b}$, $b \neq 0$, (c) $\frac{z}{z^2-4z+13}$,
(d) $\int_0^z e^{\zeta^2} d\zeta$, (e) $\int_0^z \frac{\sin\zeta}{\zeta} d\zeta$